Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

The Cubi-quadric System.

> "A familiar demonstration of the working."
> (*Love's Labor Lost*, Act I, Scene 2.)

BY J. HAMMOND.

THE CUBI-QUADRIC SYSTEM.

1. The fifteen covariants of this system are taken from Salmon's Higher Algebra (3d ed. p. 178), but the cubic is here reduced to its canonical form by writing zero for both b and c in Salmon's formulae. These are arranged according to their degree in the coefficients of the Quadric instead of their order in the variables, a uniform notation is introduced, and what must be considered (for the present purpose) simpler forms are used instead of L_4 and R.

The two notations are given connectively below.

Salmon's	The Present Notation.
u	$A_{13} = ax^3 + dy^3$
$H =$ Hessian of u	$A_{22} = adxy$
Q	$A_{33} = ad\,(ax^3 - dy^3)$
D	$A_{40} = a^2d^2$
v	$B_{02} = Ax^2 + 2Bxy + Cy^2$
L_1	$B_{11} = aCx + dAy$
Jacobian of u, v	$B_{13} = -B\,(ax^3 - dy^3) - xy\,(aCx - dAy)$
I	$B_{20} = -adB$
Jacobian of v, H	$B_{22} = ad\,(Ax^2 - Cy^2)$
L_3	$B_{31} = ad\,(aCx - dAy)$
Δ	$C_{00} = AC - B^2$
L_2	$C_{11} = B\,(aCx - dAy) - (dA^2x - aC^2y)$
$L_4 + IL_1$	$C_{31} = ad\,(dA^2x + aC^2y)$
*$L = R + 8\Delta I$	$D_{20} = a^2C^3 + d^2A^3 - 2adABC$
M	$D_{40} = ad\,(a^2C^3 - d^2A^3)$

---

* R is the resultant of the cubic u and quadric v, L the resultant of the two linears L_1 and L_2.

Here the suffixed letters show the degree in the coefficients of the Quadric, and their suffixes denote the degree in the coefficients of the Cubic and order in the variables; thus C_{31} is of the second degree in the coefficients of the Quadric, of the third degree in those of the Cubic, and linear in the variables.

2. The following syzygies may, by the use of the canonical form, be easily verified.*

No.	deg. deg. order.	No. (1), not involving groundforms in which the coefficients of the quadric appear.
(1)	0.6.6	$A_{13}^2 A_{40} - 4A_{22}^3 - A_{33}^2 = 0.$
		Nos. (2) to (7), of degree 1 in the coefficients of the quadric.
(2)	1.3.5	$A_{13}B_{22} - A_{33}B_{02} - 2A_{22}B_{13} = 0$
(3)	1.4.4	$A_{13}B_{31} - A_{33}B_{11} + 2A_{22}B_{22} = 0$
(4)	1.4.6	$A_{13}^2 B_{20} + 2A_{22}^2 B_{02} - A_{13}A_{22}B_{11} - A_{33}B_{13} = 0$
(5)	1.5.3	$A_{40}B_{13} + A_{22}B_{31} - A_{33}B_{20} = 0$
(6)	1.5.5	$A_{13}A_{40}B_{02} + 2A_{13}A_{22}B_{20} - 2A_{22}^2 B_{11} - A_{33}B_{22} = 0$
(7)	1.6.4	$A_{13}A_{40}B_{11} - 2A_{22}A_{40}B_{02} - 4A_{22}^2 B_{20} - A_{33}B_{31} = 0$
		Nos. (8) to (20), of degree 2 in the coefficients of the quadric.
(8)	2.2.4	$A_{13}C_{11} + B_{02}B_{22} + B_{11}B_{13} = 0$
(9)	2.2.6	$A_{13}^2 C_{00} - A_{13}B_{02}B_{11} + B_{02}^2 A_{22} + B_{13}^2 = 0$
(10)	2.3.3	$A_{33}C_{00} - B_{02}B_{31} + B_{13}B_{20} + A_{22}C_{11} = 0$
(11)	2.3.3	$B_{11}B_{22} - B_{02}B_{31} + 2A_{22}C_{11} = 0$
(12)	2.3.5	$A_{13}B_{02}B_{20} - 2A_{13}A_{22}C_{00} + A_{22}B_{02}B_{11} - B_{13}B_{22} = 0$
(13)	2.4.4	$A_{13}C_{31} - A_{22}B_{11}^2 - B_{22}^2 = 0$
(14)	2.4.4	$A_{33}C_{11} + B_{13}B_{31} + B_{22}^2 = 0$
(15)	2.4.4	$A_{13}C_{31} - A_{13}B_{11}B_{20} - 2A_{22}B_{02}B_{20} - A_{40}B_{02}^2 + B_{13}B_{31} = 0$
(16)	2.4.4	$A_{40}B_{02}^2 - 4A_{22}^2 C_{00} + 4A_{22}B_{02}B_{20} - B_{22}^2 = 0$
(17)	2.5.3	$2A_{22}B_{11}B_{20} - 2A_{22}C_{31} + A_{40}B_{02}B_{11} - B_{22}B_{31} = 0$

---

* Only nine of these forty-four syzygies are fundamental, in the sense in which the word is used on p. 342, Vol. VII of the *American Journal of Mathematics*. For we know that any other covariant can be expressed in the form $\varphi(A_{13}, A_{22}, A_{33}, B_{02}, B_{13}, C_{00})$, see Sylvester, Vol. I, p. 119, in this Journal. The nine fundamental syzygies are therefore those in which the remaining nine covariants appear for the first time. Or it may be simpler to say that Nos. (1) to (5) inclusive are fundamental, since from them it is possible to obtain each of the five covariants A_{40}, B_{11}, B_{22}, B_{20}, and B_{31}, in the form $\varphi(A_{13}, A_{22}, A_{33}, B_{02}, B_{13}, C_{00})$, and that Nos. (8), (13), (23), and (27) are also fundamental.

All the non-fundamental can be deduced from the fundamental, when these are known, without any direct reference to the actual values of the covariants, *e. g.* from the three fundamental syzygies Nos. (2), (3) and (8) we have

$$\begin{aligned} B_{11}(2) - B_{02}(3) + 2A_{22}(8) = {} & B_{11}(A_{13}B_{22} - A_{33}B_{02} - 2A_{22}B_{13}) \\ & - B_{02}(A_{13}B_{31} - A_{33}B_{11} + 2A_{22}B_{22}) \\ & + 2A_{22}(A_{13}C_{11} + B_{02}B_{22} + B_{11}B_{13}) = A_{13}(B_{11}B_{22} - B_{02}B_{31} + 2A_{22}C_{11}) \end{aligned}$$

from which the non-fundamental No. (11) is obtained.

(18) 2.5.3 $A_{13}A_{40}C_{00} + A_{13}B_{20}^2 - A_{22}C_{31} - B_{22}B_{31} = 0$

(19) 2.6.2 $A_{40}B_{11}^2 - 4A_{22}A_{40}C_{00} - 4A_{22}B_{20}^2 - B_{31}^2 = 0$

(20) 2.6.4 $A_{22}B_{11}B_{31} + 2A_{22}B_{20}B_{22} + A_{40}B_{02}B_{22} - A_{33}C_{31} = 0.$

No. deg. deg. order. Nos. (21) to (32), of degree 3 in the coefficients of the quadric.

(21) 3.2.4 $A_{13}B_{11}C_{00} + 2A_{22}B_{02}C_{00} - B_{02}^2B_{20} - B_{02}B_{11}^2 - B_{13}C_{11} = 0$

(22) 3.3.3 $2A_{22}B_{11}C_{00} - B_{02}B_{11}B_{20} - B_{02}C_{31} - B_{22}C_{11} = 0$

(23) 3.3.3 $A_{13}D_{20} - B_{11}^3 + 2B_{22}C_{11} + B_{02}C_{31} = 0$

(24) 3.4.2 $A_{22}D_{20} + A_{40}B_{02}C_{00} + B_{02}B_{20}^2 - B_{11}C_{31} = 0$

(25) 3.4.2 $2A_{22}D_{20} - B_{11}^2B_{20} - B_{11}C_{31} - B_{31}C_{11} = 0$

(26) 3.4.4 $A_{22}B_{11}C_{11} + 2A_{22}B_{22}C_{00} - B_{02}B_{20}B_{22} + B_{13}C_{31} = 0$

(27) 3.5.3 $A_{13}D_{40} - B_{11}^2B_{31} + B_{22}C_{31} = 0$

(28) 3.5.3 $4A_{22}B_{20}C_{11} - 2A_{22}B_{31}C_{00} + A_{40}B_{02}C_{11} + B_{11}B_{20}B_{22} + B_{22}C_{31} = 0$

(29) 3.5.3 $A_{33}D_{20} - 2A_{40}B_{13}C_{00} - B_{11}^2B_{31} - 2B_{20}^2B_{13} - B_{22}C_{31} = 0$

(30) 3.6.2 $A_{22}D_{40} + A_{40}B_{22}C_{00} + B_{20}^2B_{22} - B_{31}C_{31} = 0$

(31) 3.6.2 $2A_{22}D_{40} - A_{40}B_{11}C_{11} - B_{11}B_{20}B_{31} - B_{31}C_{31}$

(32) 3.7.3 $2A_{22}A_{40}B_{11}C_{00} + 2A_{22}B_{20}C_{31} + 2A_{22}B_{11}B_{20}^2 - A_{40}B_{11}^3 + A_{40}B_{02}C_{31} + A_{33}D_{40} = 0.$

Nos. (33) to (41), of degree 4 in the coefficients of the quadric.

(33) 4.2.2 $B_{02}D_{20} - B_{11}^2C_{00} - C_{11}^2 = 0$

(34) 4.3.3 $2A_{22}C_{00}C_{11} - B_{02}B_{20}C_{11} - B_{11}^2C_{11} - B_{11}B_{22}C_{00} - B_{13}D_{20} = 0$

(35) 4.4.2 $B_{02}D_{40} + B_{11}B_{20}C_{11} - B_{11}B_{31}C_{00} - C_{11}C_{31} = 0$

(36) 4.4.2 $B_{02}D_{40} - B_{22}D_{20} - 2C_{11}C_{31} = 0$

(37) 4.5.1 $2A_{40}C_{00}C_{11} - B_{11}D_{40} + 2B_{20}^2C_{11} + B_{31}D_{20} = 0$

(38) 4.5.3 $B_{11}B_{31}C_{11} - B_{20}B_{22}C_{11} + B_{13}D_{40} + B_{22}B_{31}C_{00} = 0$

(39) 4.6.2 $A_{40}B_{11}^2C_{00} + B_{11}^2B_{20}^2 - B_{22}D_{40} - C_{31}^2 = 0$

(40) 4.6.2 $A_{40}C_{11}^2 + 2B_{20}B_{31}C_{11} + B_{22}D_{40} - B_{31}^2C_{00} = 0$

(41) 4.7.1 $2A_{40}B_{11}B_{20}C_{00} - A_{40}B_{11}D_{20} + 2A_{40}C_{00}C_{31} + 2B_{20}^3B_{11} + 2B_{20}^2C_{31} + B_{31}D_{40} = 0.$

Nos. (42) and (43), of degree 5 in the coefficients of the quadric, and linear in the variables.

(42) 5.5.1 $2A_{40}B_{11}C_{00}^2 + 2B_{11}B_{20}^2C_{00} + B_{11}B_{20}D_{20} + C_{11}D_{40} - C_{31}D_{20} = 0$

(43) 5.7.1 $2A_{40}B_{31}C_{00}^2 + A_{40}C_{11}D_{20} - B_{11}B_{20}D_{40} + 2B_{20}^2B_{31}C_{00} + 2B_{20}B_{31}D_{20} - C_{31}D_{40} = 0.$

No. (44), invariant, of degree 6 in the coefficients of the quadric.

(44) 6.8.0 $4A_{40}^2C_{00}^3 + 8A_{40}B_{20}^2C_{00}^2 + 4A_{40}B_{20}C_{00}D_{20} - A_{40}D_{20}^2 + 4B_{20}^4C_{00} + 4B_{20}^3D_{20} + D_{40}^2 = 0.$

3. The Numerical Generating Function for the Cubi-quadric System, given by Professor Sylvester (*American Journal of Mathematics*, Vol. II, p. 295), is the fraction $\frac{N}{D}$; where

$$D = (1-b^2)(1-c^4)(1-bc^2)(1-b^3c^2)(1-bx^2)(1-c^2x^2)(1-cx^3)$$

$$\begin{aligned} N = {} & 1+c^3x^3+bcx+bcx^3+bc^2x^2+bc^3x+b^2cx+b^2c^2x^2+b^2c^3x+b^2c^4x^2+b^3c^2x^2+b^3c^4 \\ & -bc^3x^5-bc^5x^3-b^2c^3x^3-b^2c^4x^4-b^2c^5x^3-b^2c^6x^4-b^3c^4x^4-b^3c^5x^3-b^3c^6x^4 \\ & -b^3c^6x^2-b^4c^4x^2-b^4c^7x^5. \end{aligned}$$

It is not difficult to find a Real Generating Function similar to that obtained for the Quintic by Prof. Cayley in his Tenth Memoir on Quantics; viz. we have

$$(\mathrm{I})\ldots\quad \frac{\begin{array}{l} 1-A_{22}^2D_{20} \\ +A_{33}(1-D_{20}) \\ +B_{11}(1-A_{22}^2D_{20}) \\ +B_{13}(1-A_{22}) \\ +B_{22}(1-D_{20}) \\ +B_{31}(1-A_{22}) \\ +C_{11}(1-A_{22}) \\ +B_{11}^2(1-A_{22}) \\ +C_{31}(1-A_{22}) \\ +B_{11}B_{31}(1-A_{22}) \\ +B_{11}C_{11}(1-A_{22}) \\ +D_{40}(1-A_{22}) \end{array}}{(1-C_{00})(1-A_{40})(1-B_{20})(1-D_{20})(1-B_{02})(1-A_{22})(1-A_{13})} \quad \text{corresponding to} \quad \frac{\begin{array}{l} 1-b^3c^6x^4 \\ +c^3x^3-b^3c^5x^3 \\ +bcx-b^4c^7x^5 \\ +bcx^3-bc^3x^5 \\ +bc^2x^2-b^4c^4x^2 \\ +bc^3x-bc^5x^3 \\ +b^2cx-b^2c^3x^3 \\ +b^2c^2x^2-b^2c^4x^4 \\ +b^2c^3x-b^2c^5x^3 \\ +b^2c^4x^2-b^2c^6x^4 \\ +b^3c^2x^2-b^3c^4x^4 \\ +b^3c^4-b^3c^6x^2 \end{array}}{\text{Den}^{\text{r}}.\ D\ \text{as above.}}$$

Reasoning precisely similar to that employed by Prof. Cayley in the memoir just mentioned will show that each of the 12 negative terms in the numerator corresponds to a syzygy connecting terms such as $P\phi\,(C_{00}, A_{40}\cdot \ldots A_{13})$, where P is any one of the twelve positive terms of the numerator and C_{00}, A_{40}, etc. are the seven denominator forms; and that there are besides these 12 syzygies 36 others. We should therefore expect to find 48 irreducible syzygies in all; but it will be found that 4 of the 12 syzygies, represented in the numerator, are in fact reducible, so that the true number is 44, as given in the preceding article. The four reducible syzygies are those corresponding to the terms

$-A_{22}^2D_{20}$ deg. deg. order 3.6.4
$-A_{22}^2B_{11}D_{20}$ " " " 4.7.5
$-A_{22}B_{11}B_{31}$ " " " 2.6.4
$-A_{22}B_{11}C_{11}$ " " " 3.4.4

The first of these may be calculated as follows: The only compounds of deg. deg. order 3.6.4 which contain the twelve positive terms of the numerator in the first degree are found to be these

$$\begin{aligned}&A_{22}^2B_{20}C_{00},\ A_{22}^2D_{20},\ A_{22}A_{40}B_{02}C_{00},\ A_{22}B_{02}B_{20}^2,\ A_{40}B_{02}^2B_{20}\\ &B_{11}\,.\,A_{13}A_{40}C_{00},\ B_{11}\,.\,A_{13}B_{20}^2\\ &B_{11}^2\,.\,A_{22}B_{20},\ B_{11}^2\,.\,A_{40}B_{02}\\ &C_{31}\,.\,A_{13}B_{20}\,.\end{aligned}$$

Then in

$$\alpha A_{22}^2B_{20}C_{00}+\beta A_{22}^2D_{20}+\gamma A_{22}A_{40}B_{02}C_{00}+\ldots=0,$$

the values of the covariants, given in Art. 1, are substituted, and the numerical factors α, β, γ, . . . are determined by elementary algebra.

In this way we find the syzygy

$$(a)\qquad \begin{aligned}A_{22}^2D_{20}+A_{22}A_{40}B_{02}C_{00}+A_{22}B_{02}B_{20}^2+A_{13}A_{40}B_{11}C_{00}+A_{13}B_{11}B_{20}^2&\\ -2A_{22}B_{11}^2B_{20}-A_{40}B_{02}B_{11}^2=0&.\end{aligned}$$

But from Art. 2 we obtain

$$\begin{aligned}A_{22}(24)-B_{11}(17)+B_{11}(18)&=A_{22}(A_{22}D_{20}+A_{40}B_{02}C_{00}+B_{02}B_{20}^2-B_{11}C_{31})\\ &\quad-B_{11}(2A_{22}B_{11}B_{20}-2A_{22}C_{31}+A_{40}B_{02}B_{11}-B_{22}B_{31})\\ &\quad+B_{11}(A_{13}A_{40}C_{00}+A_{13}B_{20}^2-A_{22}C_{31}-B_{22}B_{31})\\ &=(a).\end{aligned}$$

The remaining three syzygies are

$$\begin{aligned}(b)\quad &2A_{13}A_{22}B_{20}D_{20}+A_{13}A_{40}B_{02}D_{20}+8A_{22}^2B_{11}B_{20}C_{00}-A_{22}^2B_{11}D_{20}+3A_{22}A_{40}B_{02}B_{11}C_{00}\\ &\quad-5A_{22}B_{02}B_{11}B_{20}^2-2A_{40}B_{02}^2B_{11}B_{20}-A_{13}A_{40}B_{11}^2C_{00}-A_{13}B_{11}^2B_{20}^2\\ &\quad-2A_{22}B_{02}B_{20}C_{31}-A_{40}B_{02}^2C_{31}=0,\\ (c)\quad &2A_{22}B_{20}B_{22}+A_{40}B_{02}B_{22}+A_{13}B_{20}B_{31}+A_{13}A_{40}C_{11}-A_{22}B_{11}B_{31}=0,\\ (d)\quad &2A_{22}B_{22}C_{00}-B_{02}B_{20}B_{22}+A_{13}B_{31}C_{00}-A_{13}B_{20}C_{11}-B_{02}B_{11}B_{31}+A_{22}B_{11}C_{11}=0.\end{aligned}$$

And the four reducing equations are

$$\begin{aligned}(a)&=A_{22}(24)-B_{11}(17)+B_{11}(18)\\ (b)&=2A_{22}B_{20}(23)+4A_{22}B_{20}(22)+A_{40}B_{02}(23)+2A_{40}B_{02}(22)-B_{11}(a)\\ (c)&=A_{40}(8)-B_{11}(5)+B_{20}(3)\\ (d)&=B_{11}(10)-B_{20}(8)+C_{00}(3),\end{aligned}$$

where the first of these has been given above and is here repeated.

4. The twelve negative terms in the numerator of the Real Generating Function have necessarily been selected in a very arbitrary manner; for it is possible to write most of the syzygies in a great many different forms, and, after we have decided on one of these forms, any term of the syzygy might be selected to represent it. Thus, leaving the other eight as they are, we may take for the four reducible syzygies

$$\begin{aligned}(a)' &= B_{11}(17) - B_{11}(18)\\ (b)' &= B_{11}(a)'\\ (c)' &= B_{11}(5)\\ (d)' &= B_{11}(10)\end{aligned}$$

which only differ from them in this, that those portions of them which are simply divisible by any of the denominator covariants have been rejected.

There is now no term $A^2_{22}D_{20}$ in $(a)'$ and no term $A^2_{22}B_{11}D_{20}$ in $(b)'$, so that these will have to be replaced by others. Moreover since (17)–(18) is properly represented by $-A_{22}C_{31}$, the proper representatives of $(a)'$ and $(b)'$ are $-A_{22}B_{11}C_{31}$ and $-A_{22}B^2_{11}C_{31}$.

The altered numerator will be

$$\begin{array}{l} 1 - A_{22}B_{11}C_{31}\\ + A_{33}(1 - D_{20})\\ + B_{11}(1 - A_{22}B_{11}C_{31})\\ + \ldots\ldots\ldots\ldots \end{array}$$

in which the remaining terms of (I) have undergone no change.

Both numerator and denominator of the Real Generating Function, after it has thus been altered, are multiplied by $1 - B_{11}$; when the transformed numerator becomes

$$\text{(II)}\ldots\quad \begin{array}{llll} 1 & - B^3_{11} - A_{22}B^2_{11} & & + A_{22}B^3_{11}\\ + A_{33} & - A_{33}B_{11} - A_{33}D_{20} & & + A_{33}B_{11}D_{20}\\ + B_{13} & - B_{11}B_{13} - A_{22}B_{13} & & + A_{22}B_{11}B_{13}\\ + B_{22} & - B_{11}B_{22} - B_{22}D_{20} & & + B_{11}B_{22}D_{20}\\ + B_{31} & - B^2_{11}B_{31} - A_{22}B_{31} & & + A_{22}B^2_{11}B_{31}\\ + C_{11} & - B^2_{11}C_{11} - A_{22}C_{11} & & + A_{22}B^2_{11}C_{11}\\ + C_{31} & - B_{11}C_{31} - A_{22}C_{31} & & + A_{22}B^3_{11}C_{31}\\ + D_{40} & - B_{11}D_{40} - A_{22}D_{40} & & + A_{22}B_{11}D_{40} \end{array}$$

The corresponding transformation of the Numerical Generating Function is obtained by simply multiplying both N and D by $1 - bcx$, when we have

$$\text{(III)}\qquad N(1 - bcx) = \begin{array}{llll} 1 & - b^3c^3x^3 & - b^2c^4x^4 & + b^3c^5x^5\\ + c^3x^3 & - bc^4x^4 & - b^3c^5x^3 & + b^4c^6x^4\\ + bcx^3 & - b^2c^2x^4 & - bc^3x^5 & + b^2c^4x^6\\ + bc^2x^2 & - b^2c^3x^3 & - b^4c^4x^2 & + b^5c^5x^3\\ + bc^3x & - b^3c^5x^3 & - bc^5x^3 & + b^3c^7x^5\\ + b^2cx & - b^4c^3x^3 & - b^2c^3x^3 & + b^4c^5x^5\\ + b^2c^3x & - b^3c^4x^2 & - b^2c^5x^3 & + b^5c^8x^6\\ + b^3c^4 & - b^4c^5x & - b^3c^6x^2 & + b^4c^7x^3 \end{array}$$

where the arrangement of the terms is the same as in (II).

The covariant B_{11} has now been transferred from the numerator into the denominator, and is henceforth to be reckoned among the denominator forms; while in consequence of this transference all the covariants, and all the syzygies, divisible by B_{11} have also disappeared from the numerator. Seven terms remain in the first column, or first positive block, of (II) or (III), all of them representing groundforms; the sixteen terms in the negative block all represent irreducible syzygies; and the existence of second syzygies is manifested by the appearance of a second positive block. There is reason to believe that one of the indicated second syzygies is compound, the remaining seven can be proved to be irreducible.

5. On account of the arbitrary selection of the negative terms in (I), previously mentioned, it is clear that, at any stage of the work, we may replace any combination of the 15 letters $A_{13} \ldots D_{40}$ by a properly chosen combination in syzygy with it. Using this principle to obtain a further transformation of II, we have

B_{11}^3 in syzygy with $A_{13}D_{20}$, No. (23)
$A_{22}B_{11}^2$ " " $A_{13}C_{31}$, No. (13)
$B_{11}^2B_{31}$ " " $A_{13}D_{40}$, No. (27)
$B_{11}^2C_{11}$ " " $B_{13}D_{20}$, No. (34)
$B_{11}B_{13}$ " " $A_{13}C_{11}$, No. (8)
$B_{11}C_{31}$ " " $A_{22}D_{20}$, No. (24)
$B_{11}D_{40}$ " " $B_{31}D_{20}$, No. (37).

The indicated substitutions may be performed so as to obtain

$$
\text{(IV)} \ldots \quad
\begin{array}{l}
\,1 - A_{13}D_{20} - A_{22}D_{20} + A_{13}A_{22}D_{20} \\
+ A_{33} - A_{33}B_{11} - A_{33}D_{20} + A_{33}B_{11}D_{20} \\
+ B_{13} - B_{13}D_{20} - A_{22}B_{13} + A_{22}B_{13}D_{20} \\
+ B_{22} - B_{11}B_{22} - B_{22}D_{20} + B_{11}B_{22}D_{20} \\
+ B_{31} - B_{31}D_{20} - A_{22}B_{31} + A_{22}B_{31}D_{20} \\
+ C_{11} - A_{13}C_{11} - A_{22}C_{11} + A_{13}A_{22}C_{11} \\
+ C_{31} - A_{13}C_{31} - A_{22}C_{31} + A_{13}A_{22}C_{31}D_{20} \\
+ D_{40} - A_{13}D_{40} - A_{22}D_{40} + A_{13}A_{22}D_{40}
\end{array}
$$

wherein some of the terms have been rearranged so that, as in (II), only denominator forms occur in the first row, and the last term of the seventh row is the last term of the first row multiplied by C_{31}. Each of the remaining six rows is divisible by a pair of denominator factors; *e. g.* the last is $D_{40}(1 - A_{13})(1 - A_{22})$.

This transformation proves that seven of the indicated second syzygies are irreducible, for (see Vol. VII, p. 337 of this Journal) any n^{th} syzygy that can be properly represented by a $(n + 1)$-ary combination of the groundforms is necessarily so. As an illustration, we have

$$\begin{array}{ll} A_{13}(11) & = A_{13}(B_{11}B_{22} - B_{02}B_{31} + 2A_{22}C_{11}) \\ -\ 2A_{22}(8) & -\ 2A_{22}(A_{13}C_{11} + B_{02}B_{22} + B_{11}B_{13}) \\ +\ B_{02}(3) & +\ B_{02}(A_{13}B_{31} - A_{33}B_{11} + 2A_{22}B_{22}) \\ -\ B_{11}(2) & -\ B_{11}(A_{13}B_{22} - A_{33}B_{02} - 2A_{22}B_{13}) = 0, \end{array}$$

which may be represented, with propriety, in several ways by a ternary combination. Thus the term $A_{22}B_{11}B_{13}$ represents it in (II), and the term $A_{13}A_{22}C_{11}$ in (IV); while (8) shows that $B_{11}B_{13}$ is in syzygy with $A_{13}C_{11}$ and forms the connecting link between the two modes of representation.

From (IV) we obtain immediately a set of names for the 44 irreducible syzygies, viz. these are the aggregate of the negative terms, and the binary combinations of the letters A_{33}, B_{13}, . . . D_{40} which form the first column. Names might of course be found with equal facility from (II), but these, though equally appropriate in other respects, would not consist exclusively of binary combinations.

In the following multiplication table each set of three numbers is the deg. deg. order of the syzygy whose name is the product of the letters standing one in the same row and the other in the same column.

	A_{33}	B_{13}	B_{22}	B_{31}	C_{11}	C_{31}	D_{40}	D_{20}
A_{33}	0.6.6							
B_{13}	1.4.6	2.2.6						
B_{22}	1.5.5	2.3.5	2.4.4					
B_{31}	1.6.4	2.4.4	2.5.3	2.6.2				
C_{11}	2.4.4	3.2.4	3.3.3	3.4.2	4.2.2			
C_{31}	2.6.4	3.4.4	3.5.3	3.6.2	4.4.2	4.6.2		
D_{40}	3.7.3	4.5.3	4.6.2	4.7.1	5.5.1	5.7.1	6.8.0	
A_{13}					2.2.4	2.4.4	3.5.3	3.3.3
A_{22}		1.3.5		1.5.3	2.3.3	2.5.3	3.6.2	3.4.2
B_{11}	1.4.4		2.3.3					
D_{20}	3.5.3	4.3.3	4.4.2	4.5.1				

The last four rows, in which the distribution seems somewhat irregular, correspond to the sixteen negative terms of (IV).

6. To obtain complete lists of the 2nd, 3rd, ... 9th syzygies, it will be necessary to multiply (IV) by each factor in succession of

$$(1-A_{33})(1-B_{13})(1-B_{22})(1-B_{31})(1-C_{11})(1-C_{31})(1-D_{40}),$$

and the form of the term $A_{13}A_{22}C_{31}D_{20}$ leads to the selection of $1-C_{31}$ for the first multiplier; for if this term represents the compound second syzygy $C_{31}(A_{13}A_{22}D_{20})$, it can be removed by this operation. To decide this question would involve some lengthy calculations, and, should the answer be in the affirmative, a precisely similar question would present itself at the next stage, and so on. In the absence of any information on this point we shall proceed on the assumption that $A_{13}A_{22}C_{31}D_{20}=C_{31}(A_{13}A_{22}D_{20})$, and similar assumptions will be made (one at each stage of the work) throughout. Or, what is the same, we shall assume that one of the terms of the last block, and one only, is removed at each multiplication. The result of the first multiplication is, however, given on both hypotheses.

Using, for the sake of brevity, P, Q, and R, to denote the groundforms, syzygies, and second syzygies represented in (IV), respectively; and writing

$$(1+P-Q+R)(1-C_{31})=1+P_1-Q_1+R_1-S_1,$$

we have in the two cases, viz.

$A_{13}A_{22}C_{31}D_{20}=C_{31}(A_{13}A_{22}D_{20})$	$A_{13}A_{22}C_{31}D_{20}$ irreducible
$P_1=P-C_{31}$	$P_1=P-C_{31}$
$Q_1=Q+C_{31}P$	$Q_1=Q+C_{31}P$
$R_1=(R-A_{13}A_{22}C_{31}D_{20})+C_{31}Q$	$R_1=R+C_{31}Q$
$S_1=(R-A_{13}A_{22}D_{20})\,C_{31}$	$S_1=C_{31}R.$

And in general, if after the n^{th} multiplication we have

$$1+P_n-Q_n+R_n-S_n+\dots$$

with $1-\mu$ for the $(n+1)^{\text{st}}$ multiplier, we shall find that

$$\begin{aligned} P_{n+1}&=P_n-\mu\\ Q_{n+1}&=Q_n+\mu P_n\\ R_{n+1}&=R_n+\mu Q_n\\ &\dots\dots\dots\dots \end{aligned}$$

where, in virtue of our assumption, the last two of these will be reduced by a single term.

Writing $\mu=1$ in the above, P_n, Q_n, R_n, ... being treated as numbers, we get the number of terms in $1+P_n-Q_n+R_n-S_n+\dots$, viz. we have

$$\begin{aligned} \Delta P_n&=-1\\ \Delta Q_n&=P_n\\ \Delta R_n&=Q_n\\ &\dots\dots \end{aligned}$$

the last two of these being reduced by unity.

The initial values obtained from (IV) by the consideration of the number of its terms are $P_0 = 7$, $Q_0 = 16$, $R_0 = 8$, $S_0 = 0$, $T_0 = 0$, etc.

Tabulating the values of P_n, Q_n, R_n, ... for different values of n, we find

n	0	1	2	3	4	5	6	7
P_n	7	6	5	4	3	2	1	0
Q_n	16	23	29	34	38	41	43	44
R_n	8	23	46	75	109	147	188	231
S_n	.	7	29	75	150	259	406	594
T_n	.	.	6	34	109	259	518	924
U_n	.	.	.	5	38	147	406	924
V_n	.	.	.	.	4	41	188	594
W_n	.	.	.	.	.	3	43	231
X_n	.	.	.	.	.	.	2	44
Y_n	.	.	.	.	.	.	.	1

and the number of terms in question is obtained by the addition of unity to the sum of the numbers in any column. The numbers in the last column show the total number of simple syzygies, 2nd syzygies, 3rd syzygies, etc., which are all of them irreducible (see *American Journal of Mathematics*, Vol. VII, p. 337, and apply the arguments used there to this case). Thus there are 44 simple syzygies, 231 second syzygies, and so on, provided only that the assumptions of the present article are legitimate; if they are not, P_n, Q_n, R_n, ... are still found by the same process, but the reduction of the last two numbers of a column by unity does not take place. It follows that if our assumptions are inadmissible the numbers in the last column must be increased.

7. Considering the successive multiplications at greater length, we have from (IV)

$$\begin{array}{ll}
R = A_{13}A_{22}D_{20} & \text{whence } S_1 = (R - A_{13}A_{22}D_{20})\, C_{31} \\
\quad + A_{33}B_{11}D_{20} & \qquad = C_{31} \times \left[\begin{array}{l} A_{33}B_{11}D_{20} \\ + A_{22}B_{13}D_{20} \\ + B_{11}B_{22}D_{20} \\ + A_{22}B_{31}D_{20} \\ + A_{13}A_{22}C_{11} \\ + A_{13}A_{22}C_{31}D_{20} \\ + A_{13}A_{22}D_{40} \end{array}\right] \\
\quad + A_{22}B_{13}D_{20} & \\
\quad + B_{11}B_{22}D_{20} & \\
\quad + A_{22}B_{31}D_{20} & \\
\quad + A_{13}A_{22}C_{11} & \\
\quad + A_{13}A_{22}C_{31}D_{20} & \\
\quad + A_{13}A_{22}D_{40} &
\end{array}$$

and six of the third syzygies represented by S_1 being certainly irreducible, the only question that arises is concerning the reducibility of $A_{13}A_{22}C_{31}^2D_{20}$.

Now observing that by No. (42) $C_{31}D_{20}$ is in syzygy with $C_{11}D_{40}$, and making this substitution in $A_{13}A_{22}C_{31}^2D_{20}$ without altering any other of the terms of S_1, we have

$$S_1 = C_{31} \times \left[\begin{array}{l} A_{33}B_{11}D_{20} \\ + A_{22}B_{13}D_{20} \\ + B_{11}B_{22}D_{20} \\ + A_{22}B_{31}D_{20} \\ + A_{13}A_{22}C_{11} \\ + A_{13}A_{22}C_{11}D_{40} \\ + A_{13}A_{22}D_{40} \end{array}\right] \qquad \begin{array}{l} T_2 = (S_1 - A_{13}A_{22}C_{11}C_{31})\,D_{40} \\ \quad = C_{31}D_{40} \times \left[\begin{array}{l} A_{33}B_{11}D_{20} \\ + A_{22}B_{13}D_{20} \\ + B_{11}B_{22}D_{20} \\ + A_{22}B_{31}D_{20} \\ + A_{13}A_{22}C_{11}D_{40} \\ + A_{13}A_{22}D_{40} \end{array}\right] \end{array}$$

where we assume, as in the first multiplication, that $A_{13}A_{22}C_{11}C_{31}D_{40}$ is reducible and $= D_{40} \times A_{13}A_{22}C_{11}C_{31}$; and where the multiplier used is $1 - D_{40}$. Observing now that the last term but one of T_2 is divisible by the last, making the assumption as to the reducibility of the corresponding 4th syzygy as before, and using the multiplier $1 - C_{11}$ we obtain

$$U_3 = (T_2 - A_{13}A_{22}C_{31}D_{40}^2)\,C_{11}$$
$$= C_{11}C_{31}D_{40}\,(A_{33}B_{11}D_{20} + A_{22}B_{13}D_{20} + B_{11}B_{22}D_{20} + A_{22}B_{31}D_{20} + A_{13}A_{22}C_{11}D_{40})$$

and, in the last term in brackets, writing $C_{31}D_{20}$ for $C_{11}D_{40}$, which is merely the reversal of the previous substitution,

$$U_3 = C_{11}C_{31}D_{20}D_{40}\,(A_{33}B_{11} + A_{22}B_{13} + B_{11}B_{22} + A_{22}B_{31} + A_{13}A_{22}C_{31}).$$

Here again, observing that $A_{13}C_{31}$ is in syzygy with $B_{13}B_{31}$, by No. (15), and making the corresponding substitution in the last term of U_3, we may write

$$U_3 = C_{11}C_{31}D_{20}D_{40}\,(A_{33}B_{11} + A_{22}B_{13} + B_{11}B_{22} + A_{22}B_{31} + A_{22}B_{13}B_{31})$$

where, after multiplying as before by $1 - B_{31}$ we obtain

$$V_4 = B_{31}C_{11}C_{31}D_{20}D_{40}\,(A_{33}B_{11} + B_{11}B_{22} + A_{22}B_{31} + A_{22}B_{13}B_{31})$$

and then using $1 - B_{13}$ for the next multiplier,

$$W_5 = B_{13}B_{31}C_{11}C_{31}D_{20}D_{40}\,(A_{33}B_{11} + B_{11}B_{22} + A_{22}B_{13}B_{31}).$$

The last term in brackets may be written $A_{22}B_{22}^2$, since No. (14) shows that $B_{13}B_{31}$ is in syzygy with B_{22}^2; and $A_{22}B_{22}^2$ itself may be further transformed into $A_{33}B_{11}B_{22}$, for $A_{22}B_{22}$ is in syzygy with $A_{33}B_{11}$ No. (3). Hence we may write

$$W_5 = B_{11}B_{13}B_{31}C_{11}C_{31}D_{20}D_{40}\,(A_{33} + B_{22} + A_{33}B_{22}).$$

Finally multiplying by $1 - B_{22}$ and $1 - A_{33}$ we obtain successively

$$X_6 = B_{11}B_{13}B_{22}B_{31}C_{11}C_{31}D_{20}D_{40}\,(B_{22} + A_{33}B_{22})$$

and

$$Y_7 = A_{33}^2B_{11}B_{13}B_{22}^2B_{31}C_{11}C_{31}D_{20}D_{40}.$$

In the course of the work the following reductions have been assumed.

			Deg. Deg. Order.
That of the 2nd	syzygy	$A_{13}A_{22}C_{31}D_{20}$	5.8.6
3rd	"	$A_{13}A_{22}C_{11}C_{31}D_{40}$	7.11.7
4th	"	$A_{13}A_{22}C_{11}C_{31}D_{40}^2$	10.15.7
5th	"	$A_{22}B_{13}B_{31}C_{11}C_{31}D_{20}D_{40}$	12.16.8
6th	"	$A_{22}B_{13}B_{31}^2C_{11}C_{31}D_{20}D_{40}$	13.19.9
7th	"	$A_{33}B_{11}B_{22}B_{13}B_{31}C_{11}C_{31}D_{20}D_{40}$	14.20.12
8th	"	$A_{33}B_{11}B_{13}B_{22}^2B_{31}C_{11}C_{31}D_{20}D_{40}$	15.22.14

In every other case each n^{th} syzygy is represented by a $(n+1)$-ary combination of the groundforms, and is therefore irreducible. The case of Y_7 is only an apparent exception to this rule, for (Art. 2, Nos. 13 to 16) $A_{33}C_{11}$ is in syzygy with $A_{13}C_{31}$; so that we may write

$$Y_7 = A_{13}A_{33}B_{11}B_{13}B_{22}^2B_{31}C_{31}^2D_{20}D_{40}.$$

And since $B_{22}B_{31}$ is in syzygy with $A_{22}C_{31}$ (Art. 2, Nos. 17, 18),

$$Y_7 = A_{13}A_{22}A_{33}B_{11}B_{13}B_{22}C_{31}^3D_{20}D_{40}.$$

Lastly, since $A_{13}A_{22}B_{11}$ is in syzygy with $A_{33}B_{13}$ (Art. 2, No. 4),

$$Y_7 = A_{33}^2B_{13}^2B_{22}C_{31}^3D_{20}D_{40},$$

which is the product of 10 letters only, and may properly be taken as a name for the unique 9th syzygy of deg. deg. order 15.25.17.

Can a similar transformation be effected in the case of any one of the 7 assumed compounds? Certainly not for the first of them, for there is no ternary product of the groundforms of deg. deg. order 5.8.6. It follows then that if the 2nd syzygy $A_{13}A_{22}C_{31}D_{20}$ is in fact irreducible, and this would make all the rest so, the proof must proceed in some other way than the transformation of its name into a ternary product.

8. Proceeding now to the formation of a catalogue, *i. e.* a list with names, of the second syzygies, we have directly from (IV),

$$\begin{array}{lll}
P = A_{33} & Q = A_{13}D_{20} + A_{22}D_{20} & R - A_{13}A_{22}C_{31}D_{20} = A_{13}A_{22}D_{20} \\
\quad + B_{13} & \quad + A_{33}B_{11} + A_{33}D_{20} & \quad + A_{33}B_{11}D_{20} \\
\quad + B_{22} & \quad + B_{13}D_{20} + A_{22}B_{13} & \quad + A_{22}B_{13}D_{20} \\
\quad + B_{31} & \quad + B_{11}B_{22} + B_{22}D_{20} & \quad + B_{11}B_{22}D_{20} \\
\quad + C_{11} & \quad + B_{31}D_{20} + A_{22}B_{31} & \quad + A_{22}B_{31}D_{20} \\
\quad + C_{31} & \quad + A_{13}C_{11} + A_{22}C_{11} & \quad + A_{13}A_{22}C_{11} \\
\quad + D_{40} & \quad + A_{13}C_{31} + A_{22}C_{31} & \quad + A_{13}A_{22}D_{40} \\
 & \quad + A_{13}D_{40} + A_{22}D_{40} &
\end{array}$$

where the last column forms a portion of the catalogue.

The remaining portion is, with the notation of the preceding article, by means of the equations

$$
\begin{aligned}
R_1 &= (R - A_{13}A_{22}C_{31}D_{20}) + C_{31}Q \\
R_2 &= R_1 + D_{40}Q_1 \\
R_3 &= R_2 + C_{11}Q_2 \\
R_4 &= R_3 + B_{31}Q_3 \\
R_5 &= R_4 + B_{13}Q_4 \\
R_6 &= R_5 + B_{22}Q_5 \\
R_7 &= R_6 + A_{33}Q_6
\end{aligned}
$$

found to be the additional part of R_7

$$= C_{31}Q + D_{40}Q_1 + C_{11}Q_2 + B_{31}Q_3 + B_{13}Q_4 + B_{22}Q_5 + A_{33}Q_6 .$$

In which, since

$$
\begin{aligned}
Q_1 &= Q + C_{31}P \\
Q_2 &= Q_1 + D_{40}P_1 \\
Q_3 &= Q_2 + C_{11}P_2 \\
Q_4 &= Q_3 + B_{31}P_3 \\
Q_5 &= Q_4 + B_{13}P_4 \\
Q_6 &= Q_5 + B_{22}P_5
\end{aligned}
$$

we may substitute these values, beginning with the last, and find for the remaining portion of the catalogue the expression,

$$
\begin{aligned}
&Q\,(C_{31} + D_{40} + C_{11} + B_{31} + B_{13} + B_{22} + A_{33}) + (D_{40} + C_{11} + B_{31} + B_{13} + B_{22} + A_{33})\,C_{31}P \\
&+ (C_{11} + B_{31} + B_{13} + B_{22} + A_{33})\,D_{40}P_1 + (B_{31} + B_{13} + B_{22} + A_{33})\,C_{11}P_2 \\
&+ (B_{13} + B_{22} + A_{33})\,B_{31}P_3 + (B_{22} + A_{33})\,B_{13}P_4 + A_{33}B_{22}P_5 .
\end{aligned}
$$

In this we substitute for P and Q their given values, writing at the same time,

$$
\begin{aligned}
P_1 &= P - C_{31} = A_{33} + B_{13} + B_{22} + B_{31} + C_{11} + D_{40} \\
P_2 &= P_1 - D_{40} = A_{33} + B_{13} + B_{22} + B_{31} + C_{11} \\
P_3 &= P_2 - C_{11} = A_{33} + B_{13} + B_{22} + B_{31} \\
P_4 &= P_3 - B_{31} = A_{33} + B_{13} + B_{22} \\
P_5 &= P_4 - B_{13} = A_{33} + B_{22}
\end{aligned}
$$

then multiply out, add in the given value of $R - A_{13}A_{22}C_{31}D_{20}$, and arrange the terms in any convenient manner. In the arrangement adopted, terms, *i. e.* names of second syzygies, of the same deg. deg. order are written beneath it, and their frequency below them: thus

$$
\begin{gathered}
1.6.8 \\
A_{22}A_{33}B_{13} \\
(1)
\end{gathered}
$$

is understood to mean that there is one second syzygy, of deg. deg. order 1.6.8, whose name is $A_{22}A_{33}B_{13}$.

9. Catalogue of the Second Syzygies:

1.6.8	1.7.7	1.7.9	1.8.6	1.8.8	1.9.7
$A_{22}A_{33}B_{13}$	$A^2_{33}B_{11}$	$A^2_{33}B_{13}$	$A_{22}A_{33}B_{31}$	$A^2_{33}B_{22}$	$A^2_{33}B_{31}$
(1)	(1)	(1)	(1)	(1)	(1)

2.4.6	2.4.8	2.5.7	2.5.9	2.6.6	2.6.8
$A_{13}A_{22}C_{11}$	$A_{22}B^2_{13}$	$A_{22}B_{13}B_{22}$ $A_{33}B_{11}B_{13}$ $A_{13}A_{33}C_{11}$ . .	$A_{33}B^2_{13}$	$A_{22}B_{13}B_{31}$ $A_{22}A_{33}C_{11}$ $A_{22}B_{31}B_{13}$ $A_{33}B_{11}B_{22}$ $B_{11}B_{22}A_{33}$	$A_{33}B_{13}B_{22}$ $B_{13}B_{22}A_{33}$ . . .
(1)	(1)	(3)	(1)	(5)	(2)

2.7.5	2.7.7	2.8.4	2.8.6	2.9.5	2.9.7
$A_{22}B_{22}B_{31}$ $A_{33}B_{11}B_{31}$ . . .	$A_{13}A_{33}C_{31}$ $A^2_{33}C_{11}$ $A_{33}B_{13}B_{31}$ $A_{33}B_{31}B_{13}$ $A_{33}B^2_{22}$	$A_{22}B^2_{31}$	$A_{22}A_{33}C_{31}$ $A_{33}B_{22}B_{31}$ $A_{33}B_{31}B_{22}$ . .	$A_{33}B^2_{31}$	$A^2_{33}C_{31}$ . . . .
(2)	(5)	(1)	(3)	(1)	(1)

3.3.7	3.4.6	3.4.8	3.5.5	3.5.7	3.6.4
$A_{13}B_{13}C_{11}$	$A_{13}B_{22}C_{11}$ $A_{22}B_{13}C_{11}$ $A_{22}C_{11}B_{13}$ $B_{11}B_{13}B_{22}$.	$B_{22}B^2_{13}$	$A_{13}A_{22}D_{20}$ $A_{22}B_{22}C_{11}$ $A_{33}B_{11}C_{11}$ $A_{13}B_{31}C_{11}$ $B_{11}B^2_{22}$	$A_{13}B_{13}C_{31}$ $A_{33}B_{13}C_{11}$ $A_{33}C_{11}B_{13}$ $B_{13}B^2_{22}$ $B^2_{13}B_{31}$	$A_{22}B_{31}C_{11}$ $A_{22}C_{11}B_{31}$ $B_{11}B_{22}B_{31}$ . .
(1)	(4)	(1)	(5)	(5)	(3)

3.6.6	3.7.5	3.7.7	3.8.4	3.8.6	3.9.5	3.10.6
$A_{13}A_{33}D_{20}$ $A_{13}B_{22}C_{31}$ $A_{22}B_{13}C_{31}$ $A_{22}C_{31}B_{13}$ $A_{33}B_{22}C_{11}$ $A_{33}C_{11}B_{22}$ $B_{13}B_{22}B_{31}$ $B_{13}B_{31}B_{22}$.	$A_{13}A_{22}D_{40}$ $A_{22}A_{33}D_{20}$ $A_{13}B_{31}C_{31}$ $A_{22}B_{22}C_{31}$ $A_{33}B_{11}C_{31}$ $A_{33}B_{31}C_{11}$ $A_{33}C_{11}B_{31}$ $B_{13}B^2_{31}$ $B^2_{22}B_{31}$	$A_{33}B_{13}C_{31}$ $A_{33}C_{31}B_{13}$	$A_{22}B_{31}C_{31}$ $A_{22}C_{31}B_{31}$ $B_{22}B^2_{31}$	$A_{13}A_{33}D_{40}$ $A^2_{33}D_{20}$ $A_{33}B_{22}C_{31}$ $A_{33}C_{31}B_{22}$	$A_{22}A_{33}D_{40}$ $A_{33}B_{31}C_{31}$ $A_{33}C_{31}B_{31}$	$A^2_{33}D_{40}$ . . . .
(8)	(9)	(2)	(3)	(4)	(3)	(1)

4.3.5	4.3.7	4.4.4	4.4.6	4.5.5	4.5.7
$A_{13}C_{11}^2$	$B_{13}^2C_{11}$	$A_{22}C_{11}^2$	$A_{13}B_{13}D_{20}$	$A_{13}B_{22}D_{20}$	$B_{13}^2C_{31}$
.	.	$B_{11}B_{22}C_{11}$	$B_{13}B_{22}C_{11}$	$A_{13}C_{11}C_{31}$	.
.	.	.	$B_{13}C_{11}B_{22}$	$A_{13}C_{31}C_{11}$	.
.	.	.	.	$A_{22}B_{13}D_{20}$	.
.	.	.	.	$A_{22}D_{20}B_{13}$	.
.	.	.	.	$A_{33}C_{11}^2$	.
.	.	.	.	$B_{13}B_{31}C_{11}$	.
.	.	.	.	$B_{13}C_{11}B_{31}$	.
.	.	.	.	$B_{22}^2C_{11}$	.
(1)	(1)	(2)	(3)	(9)	(1)

4.6.4	4.6.6	4.7.3	4.7.5	4.8.4	4.8.6
$A_{13}B_{31}D_{20}$	$A_{13}B_{13}D_{40}$	$A_{22}B_{31}D_{20}$	$A_{13}B_{22}D_{40}$	$A_{13}B_{31}D_{40}$	$A_{33}B_{13}D_{40}$
$A_{33}B_{11}D_{20}$	$A_{33}B_{13}D_{20}$	$A_{22}D_{20}B_{31}$	$A_{13}C_{31}^2$	$A_{22}B_{22}D_{40}$	$A_{33}D_{40}B_{13}$
$A_{22}B_{22}D_{20}$	$A_{33}D_{20}B_{13}$	$B_{31}^2C_{11}$	$A_{22}B_{13}D_{40}$	$A_{22}C_{31}^2$	.
$A_{22}C_{11}C_{31}$	$B_{13}B_{22}C_{31}$	.	$A_{22}D_{40}B_{13}$	$A_{33}B_{11}D_{40}$	.
$A_{22}C_{31}C_{11}$	$B_{13}C_{31}B_{22}$	.	$A_{33}B_{22}D_{20}$	$A_{33}B_{31}D_{20}$	.
$B_{11}B_{22}C_{31}$	.	.	$A_{33}D_{20}B_{22}$	$A_{33}D_{20}B_{31}$	.
$B_{22}B_{31}C_{11}$	.	.	$A_{33}C_{11}C_{31}$	$B_{22}B_{31}C_{31}$	.
$B_{22}C_{11}B_{31}$	.	.	$A_{33}C_{31}C_{11}$	$B_{22}C_{31}B_{31}$	.
.	.	.	$B_{13}B_{31}C_{31}$	.	.
.	.	.	$B_{13}C_{31}B_{31}$	.	.
.	.	.	$B_{22}^2C_{31}$	.	.
(8)	(5)	(3)	(11)	(8)	(2)

4.9.3	4.9.5	4.10.4
$A_{22}B_{31}D_{40}$	$A_{33}C_{31}^2$	$A_{33}B_{31}D_{40}$
$A_{22}D_{40}B_{31}$	$A_{33}B_{22}D_{40}$	$A_{33}D_{40}B_{31}$
$B_{31}^2C_{31}$	$A_{33}D_{40}B_{22}$	.
(3)	(3)	(2)

5.3.5	5.4.4	5.4.6	5.5.3	5.5.5	5.6.4	5.6,6
$B_{13}C_{12}^1$	$A_{13}C_{11}D_{20}$	$B_{13}^2D_{20}$	$A_{22}C_{11}D_{20}$	$B_{13}B_{22}D_{20}$	$A_{13}C_{11}D_{40}$	$B_{13}^2D_{40}$
.	$B_{22}C_{11}^2$	.	$B_{11}B_{22}D_{20}$	$B_{13}D_{20}B_{22}$	$A_{13}C_{31}D_{20}$	.
.	.	.	$B_{31}C_{11}^2$	$B_{13}C_{11}C_{31}$	$A_{13}D_{40}C_{11}$	.
.	.	.	.	$B_{13}C_{31}C_{11}$	$A_{33}C_{11}D_{20}$	.
.	.	.	.	.	$B_{13}B_{31}D_{20}$	.
.	.	.	.	.	$B_{13}D_{20}B_{31}$	.
.	.	.	.	.	$B_{22}^2D_{20}$	.
.	.	.	.	.	$B_{22}C_{11}C_{31}$	.
.	.	.	.	.	$B_{22}C_{31}C_{11}$	.
(1)	(2)	(1)	(3)	(4)	(9)	(1)

5.7.3	5.7.5	5.8.2	5.8.4	5.9.3	5.10.2	5.10.4
$A_{22}C_{11}D_{40}$	$B_{13}B_{22}D_{40}$	$B_{31}^2D_{20}$	$A_{13}C_{31}D_{40}$	$A_{22}C_{31}D_{40}$	$B_{31}^2D_{40}$	$A_{33}C_{31}D_{40}$
$A_{22}D_{40}C_{11}$	$B_{13}D_{40}B_{22}$	.	$A_{13}D_{40}C_{31}$	$A_{22}D_{40}C_{31}$	.	$A_{33}D_{40}C_{31}$
$A_{22}C_{31}D_{20}$	$B_{13}C_{31}^2$	.	$A_{33}C_{11}D_{40}$	$B_{22}B_{31}D_{40}$	.	.
$B_{11}B_{22}D_{40}$	.	.	$A_{33}D_{40}C_{11}$	$B_{22}D_{40}B_{31}$	.	.
$B_{22}B_{31}D_{20}$	.	.	$A_{33}C_{31}D_{20}$	$B_{31}C_{31}^2$	.	.
$B_{22}D_{20}B_{31}$	.	.	$B_{13}B_{31}D_{40}$	.	.	.
$B_{31}C_{11}C_{31}$	.	.	$B_{13}D_{40}B_{31}$	.	.	.
$B_{31}C_{11}C_{31}$	.	.	$B_{22}^2D_{40}$	.	.	.
.	.	.	$B_{22}C_{31}^2$	.	.	.
(8)	(3)	(1)	(9)	(5)	(1)	(2)

6.4.4	6.5.3	6.6.2	6.6.4	6.7.3	6.8.2
$B_{13}C_{11}D_{20}$	$B_{22}C_{11}D_{20}$	$B_{31}C_{11}D_{20}$	$B_{13}C_{31}D_{20}$	$A_{13}D_{20}D_{40}$	$A_{22}D_{20}D_{40}$
.	$C_{11}^2C_{31}$	.	$B_{13}C_{11}D_{40}$	$B_{22}C_{31}D_{20}$	$B_{31}C_{31}D_{20}$
.	.	.	$B_{13}D_{40}C_{11}$	$B_{22}C_{11}D_{40}$	$B_{31}C_{11}D_{40}$
.	.	.	.	$B_{22}D_{40}C_{11}$	$B_{31}D_{40}C_{11}$
.	.	.	.	$C_{11}C_{31}^2$	.
(1)	(2)	(1)	(3)	(5)	(4)

6.8.4	6.9.3	6.10.2	6.11.3
$B_{13}C_{31}D_{40}$	$A_{13}D_{40}^2$	$A_{22}D_{40}^2$	$A_{33}D_{40}^2$
$B_{13}D_{40}C_{31}$	$A_{33}D_{20}D_{40}$	$B_{31}C_{31}D_{40}$	.
.	$B_{22}C_{31}D_{40}$	$B_{31}D_{40}C_{31}$	.
.	$B_{22}D_{40}C_{31}$	.	.
(2)	(4)	(3)	(1)

7.6.2	7.7.3	7.8.2	7.9.1	7.9.3	7.10.2	7.11.1
$C_{11}^2D_{40}$	$B_{13}D_{20}D_{40}$	$B_{22}D_{20}D_{40}$	$B_{31}D_{20}D_{40}$	$B_{13}D_{40}^2$	$B_{22}D_{40}^2$	$B_{31}D_{40}^2$
.	.	$C_{11}C_{31}D_{40}$	.	.	$C_{31}^2D_{40}$	.
.	.	$C_{11}D_{40}C_{31}$	.	.	.	.
(1)	(1)	(3)	(1)	(1)	(2)	(1)

8.9.1	8.11.1
$C_{11}D_{40}^2$	$C_{31}D_{40}^2$
(1)	(1)

10. It would be tedious to give an enumeration of the irreducible 3rd, 4th... syzygies, especially so since no new principle would be illustrated thereby. Besides, it would be very desirable to previously ascertain whether there is an irreducible second syzygy of deg. deg. order 5.8.6 or not. As this would also require a lengthy investigation, I close the present paper with a few remarks in conclusion.

First. If it should eventually be proved that there is an irreducible second syzygy of deg. deg. order 5.8.6, the catalogue would be the same as before, but the name $A_{13}A_{22}C_{31}D_{20}$ would have to be added to the 231 names contained in it.

Secondly. An inspection of the deg. deg. orders will show that there is no case of the coexistence of an irreducible simple syzygy with an irreducible second syzygy of the same deg. deg. order; but it would be unsafe to assume that this is so for every quantic or system of quantics. The case of the Quintic itself shows that this is not so; for the Quintic has both a simple syzygy and a second syzygy of deg. order 8.16.

Thirdly. If, as I have always found to be the case, simple syzygies may be represented by binary combinations and second syzygies by ternary combinations of the groundforms; there is an advantage in working with the Real Generating Function instead of the Numerical. For two terms, one of which is a binary and the other a ternary combination of the groundforms cannot cancel each other; but if, as in the case when the Numerical Generating Function is used, both are represented by the same symbol, they may so cancel each other. This advantage is of course limited to those cases in which irreducible first and second, or n^{th} and $(n+1)^{\text{st}}$ syzygies coexist; but the Real Generating Function gives names as well as the frequency of syzygies, whereas the Numerical Generating Function gives the frequency only, and this constitutes a special advantage in the use of the former over that of the latter.

Finally, the name of a second syzygy being PQR, where each letter denotes one of the groundforms, there cannot be more than two second syzygies of this name when P, Q, and R are three distinct letters, there cannot be more than one second syzygy of this name when P, Q, R are only two distinct letters (*i. e.* there cannot be more than one whose name is PQ^2), and there can be no second syzygy whose name contains only one distinct letter (P^3). This law is revealed by an inspection of the names in the catalogue, its significance is best seen by the following example.

Name $A_{33}^2 B_{11}$, deg. deg. order 1.7.7

$A_{33}(A_{13}B_{31} - A_{33}B_{11} + 2A_{22}B_{22})$ outside factor A_{33}, inside factor $A_{33}B_{11}$
$- B_{11}(A_{13}^2 A_{40} - 4A_{22}^3 - A_{33}^2)$ " " B_{11} " " A_{33}^2
$+ 2A_{22}(A_{13}A_{40}B_{02} + 2A_{13}A_{22}B_{20} - 2A_{22}^2 B_{11} - A_{33}B_{22})$
$+ A_{13}(A_{13}A_{40}B_{11} - 2A_{22}A_{40}B_{02} - 4A_{22}^2 B_{20} - A_{33}B_{31}) = 0$.

Observe that it is necessary to the existence of a second syzygy that its name should be divisible in two ways, into an outside and an inside factor, the outside factor being the name of a covariant and the inside factor the name of a first syzygy: for the name P^2Q, the only two possible ways are P outside, PQ inside, and Q outside, P^2 inside. Thus there will be one second syzygy, and only one, whose name is P^2Q, if both P^2 and PQ are the names of syzygies. But if there is no syzygy named P^2, or no syzygy named PQ, there will be no second syzygy named P^2Q. Again, P^3 can only be divided in one way, viz. P outside and P^2 inside: hence, there is no second syzygy named P^3.

For the second syzygy named PQR, we have the three divisions (1) P outside, QR inside; (2) Q outside, PR inside; (3) R outside, PQ inside. Suppose now that there exist three syzygies named QR, PR, and PQ. These may be written in the form $QR = \ldots$, $PR = \ldots$, and $PQ = \ldots$, in which the name of any one of them does not appear in the other two; and then we should have three second syzygies, viz.

$$
\begin{aligned}
P(QR\ldots) - Q(PR\ldots) + \ldots &= 0\\
Q(PR\ldots) - R(PQ\ldots) + \ldots &= 0\\
R(PQ\ldots) - P(QR\ldots) + \ldots &= 0
\end{aligned}
$$

where the terms written down may be called the distinctive portions of the second syzygies. But the distinctive portion of the third of these is the sum of the distinctive portions of the other two; proving that there are not more than two second syzygies named PQR. If PQ should not be the name of a syzygy the last two "second syzygies" would be non-existent, and there would remain only the first. The example given above illustrates this case also if we alter the name to $A_{13}A_{33}B_{31}$ (as we may if we choose). For there is no syzygy that can be named $A_{13}A_{33}$.

BUCKHURST HILL, ESSEX, Oct. 3, 1885.